Vibration Analysis of Cross-Ply Plates Under Initial Stress Using Refined Theory

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Abstract

Natural frequency under initial stresses for simply supported cross-ply composite laminated plates (E glass- fiber) are obtained using Refind theory (RPT). This theory accounts for parabolic distribution of the transverse shear strain through the plate thickness and satisfies the zero traction boundary conditions on the surfaces of the plate without using shear correction factors. The governing equations for Eigen value problem under initial stress are derived using Hamilton's principle and solved using Navier solution for simply supported cross-ply symmetric and antisymmetric laminated plates. The effect of many design factors such as modulus ratio, thickness ratio and number of laminates on the Natural frequency and buckling stresses of orthotropic plates are studied. The results are compared with other researcher.

Keyword: Composite laminated plate, buckling analysis, free vibration analysis, Refined plate theory.

1. Introduction

Laminated composite plates have very importance in the engineering applications because of their useful features so a many variety of laminated theories for static and dynamic behavior have been developed such as approximate, experimental and exact methods.

[1] presented static analysis using higher-order refined theory of angle ply plate and sandwich plates hitherto. No requirement to use shear correction factors (SCF), because the transverse-shear strains vary parabolically from side to side which lead to vanish the shear-stresses on the upper and bottom surface of the plate. From principle of potential energy, the equations of equilibrium are derived and solved by using Navier-type method. Correctness of the theoretical preparations and the solution method confirmed by comparing the results with other theory described in the literature.

[2] Presented buckling analysis of SS plate exposed to in-plane loading using refined plate theory of orthotropic and isotropic plates. The governing equations G.E which derivative from the principle of virtul-displacements, and solved by using the Navier method. This theory is simple, comparable to the(FSDT) theory and there no exists a need for using SCF. [3] Studied a two-variable Refind theory (RPT) of laminated composite plates. The theory contents the zero traction B.C on the upper and bottom faces of the plate without wanting to use SCF. The equations of motion are derivative using Hamilton’s principle (H.P) and solved using Naveir method of angle-ply and cross-ply antisymmetric laminate. This theory is simple and accurate in solving the buckling behaviors and static bending of laminated composite plates. [4] Studied free vibration of laminated composite plates using two variable Refined plate theory (RPT) and using Hamilton’s principle to derive the equations of motion, and these equations solved using Naveir solutions of cross-ply and angle-ply antisymmetric laminates. This theory is accurate and effective in obtain the natural frequencies N.F of laminated composite plates. [5] Studied the buckling analysis using Refind theory for orthotropic plates. No requirement to use SCF in this theory and the Governing equations solved...
using Levy-type method. It considering the effect of some design limitations such as boundary conditions, orthotropy ratio, thickness ratio and loading condition on the critical-buckling load. [6] Presented free vibration investigation of functionally arranged material (FGM) sandwich rectangular plates by the four variable refined theory (RPT) which not requirement to use SCF. The equation of motion achieved using Hamilton’s principle for the (FGM) sandwich plates and these equations solved by using the Navier type. This theory simple and accurate in resolving the free-vibration behavior of the functionally arranged material sandwich plates when its results compared with other theories such as classical laminated theory (CLP), first order theory (FSDT). [7] Presented free vibration analysis of simply supported plate which made of functionally arranged materials using four variable Refind theory. No requirement to use shear correction factors, because the transverse-shear strains vary parabolically from side to side the thickness which lead to disappear the shear stresses on the upper and bottom faces of the plate. From the principle of virtual displacements, the governing equations for the (FGM) rectangular plates are derived and solved by using Navier-type method. The natural frequencies are found using the Ritz method in the case of FG clamped plates. The strength of this present theory which gave accurate free vibration of FG plate shown by comparing the present results with others theories and also the influence of vining rises, aspect ratios, and thick ratio on the free-vibration of the FG plates is showed. [8] Presented free vibration analysis of rectangular plate with two opposite edges simply supported (SS) and the other two edges having arbitrary boundary conditions using ‘refined plate theory’. From the principle of virtual displacements, the governing equations are derived and solved by using the Levy-type method. No need to use shear correction factors in this theory, it considering the effect of some design parameters such as boundary conditions, modulus ratio, and aspect ratio on the natural frequency.

In present work, the equation of motion of Refined plate theory simple and accurate in resolving the free-vibration for simply supported plates using Navier solution.

2. Theoretical Analysis
2.1 Displacement Field

In present work, a rectangular plate of total thickness (h) of (n) orthotropic layers with the coordinate system as shown in Fig (1) are considered the displacement of Refined plate theory (RPT) which satisfies equilibrium conditions at the top and bottom faces of the plate without using shear correction factor is developed. The transverse displacement W contains three components; bending $w_{be}$, extension $w_{a}$ and shear $w_{sh}$ which these components are functions of coordinates x, y, and time t only. Similarly, the displacements u in x-direction and v in y-direction have bending, extension, and shear components [3].

$U = u + u_{be} + u_{sh}$
$V = v + v_{be} + v_{sh}$

$W(x,y,z,t) = w_{a}(x,y,t) + w_{be}(x,y,t) + w_{sh}(x,y,t)$

The shear components $w_{sh}$ and $v_{sh}$, $w_{sh}$ lead to the parabolic variations of shear strains $\gamma_{xz}$, $\gamma_{yz}$ and to shear stresses $\sigma_{xz}$, $\sigma_{yz}$ through the thickness of the plate in such a way that shear stresses $\sigma_{xz}$, $\sigma_{yz}$ are zero at the bottom and top surfaces of the plate.

$u_{sh} = z \left[ \frac{1}{4} - \frac{5}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial w_{sh}}{\partial x}$
$v_{sh} = z \left[ \frac{1}{4} - \frac{5}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial w_{sh}}{\partial y}$

The following displacement field assumptions [3]:

$U(x,y,z,t) = u(x,y,t) - z \left[ \frac{\partial w_{be}}{\partial x} + z \left[ \frac{1}{4} - \frac{5}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial w_{sh}}{\partial x} \right]$ + $\frac{5}{3} \left( \frac{z}{h} \right)^2 \frac{\partial w_{sh}}{\partial x}$
$V(x,y,z,t) = v(x,y,t) - z \left[ \frac{\partial w_{be}}{\partial y} + z \left[ \frac{1}{4} - \frac{5}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial w_{sh}}{\partial y} \right]$ $\frac{5}{3} \left( \frac{z}{h} \right)^2 \frac{\partial w_{sh}}{\partial y}$
$W(x,y,z,t) = w_{a}(x,y,t) + w_{be}(x,y,t)$ + $w_{sh}(x,y,t)$

... (1)

For small strain, the strain-displacement relations take the form:

$\varepsilon_{xx} = \frac{\partial u}{\partial x}$
$\varepsilon_{yy} = \frac{\partial v}{\partial y}$
$\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$ $- \frac{1}{2} \gamma_{xy}$
$\varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$ $- \frac{1}{2} \gamma_{xz}$
$\varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$ $- \frac{1}{2} \gamma_{yz}$

... (2)

By substituting eq. (1) into eq. (2) to give:

$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z \frac{\partial ^2 w_{be}}{\partial x^2} + z \left[ \frac{1}{4} - \frac{5}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial ^2 w_{sh}}{\partial x^2} + \frac{5}{3} \left( \frac{z}{h} \right)^2 \frac{\partial w_{sh}}{\partial x}$
$\varepsilon_{yy} = \frac{\partial v}{\partial y} - z \frac{\partial ^2 w_{be}}{\partial y^2} + z \left[ \frac{1}{4} - \frac{5}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial ^2 w_{sh}}{\partial y^2} + \frac{5}{3} \left( \frac{z}{h} \right)^2 \frac{\partial w_{sh}}{\partial y}$
\[ y_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w_{be}}{\partial x \partial y} + 2z \left( \frac{z}{h} \right)^2 \]
\[ \frac{5}{3} \left( \frac{h}{2} \right)^2 \frac{\partial^2 w_{sh}}{\partial y \partial z} \]
\[ y_{yz} = \frac{\partial^2 w_{sh}}{\partial y \partial z} + \frac{5}{3} \left( \frac{h}{2} \right)^2 \frac{\partial w_{sh}}{\partial y} \]
\[ y_{xz} = \frac{\partial w_{sh}}{\partial x} + \frac{5}{3} \left( \frac{h}{2} \right)^2 \frac{\partial w_{sh}}{\partial y} \] ... (3)

The strain field is:
\[
\begin{align*}
\{ \varepsilon_x \} &= \left\{ \varepsilon_x^0 \right\} + f \left\{ k_x^0 \right\}, \\
\{ \varepsilon_y \} &= \left\{ \varepsilon_y^0 \right\} + g \left\{ \varepsilon_y^0 \right\}, \\
\{ \varepsilon_{xy} \} &= \left\{ \varepsilon_{xy} \right\} + f \left\{ k_{xy}^0 \right\}, \\
\{ \varepsilon_{xz} \} &= \left\{ \varepsilon_{xz} \right\} + g \left\{ \varepsilon_{xz} \right\}, \\
\{ \varepsilon_{yz} \} &= \left\{ \varepsilon_{yz} \right\} + f \left\{ k_{yz}^0 \right\}, \\
\{ \varepsilon_{y} \} &= \left\{ \varepsilon_{y} \right\} + g \left\{ \varepsilon_{y} \right\}
\end{align*}
\]
\[
\text{Where:} \\
\begin{align*}
\{ \varepsilon_x^0 \} &= \left\{ \frac{\partial u}{\partial x} \right\}, \\
\{ \varepsilon_y^0 \} &= \left\{ \frac{\partial v}{\partial y} \right\}, \\
\{ \varepsilon_{xy} \} &= \left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\}.
\end{align*}
\]
\[ y_{xz} = \frac{\partial w_{a}}{\partial x} + \frac{5}{3} \left( \frac{h}{2} \right)^2 \frac{\partial w_{sh}}{\partial y} \]
\[ y_{yz} = \frac{\partial w_{a}}{\partial y} + \frac{5}{3} \left( \frac{h}{2} \right)^2 \frac{\partial w_{sh}}{\partial y} \]
\[ y_{xz} = \frac{\partial w_{a}}{\partial x} + \frac{5}{3} \left( \frac{h}{2} \right)^2 \frac{\partial w_{sh}}{\partial y} \]
\[ f = -\frac{1}{2} \frac{z}{h} + \frac{5}{2} \frac{z^2}{h^2}, \quad g = \frac{5}{3} \frac{z}{h} - 5 \left( \frac{z}{h} \right)^2 \] ... (5)

2.2 Principle of Virtual Work

Using Hamilton’s principle, the equations of motion of the refined plate theory will be derived.

Reddy, 2004
\[ 0 = \int_0^h (\delta U + \delta V - \delta T) \, dt \] ... (6)

- The virtual strain energy \( \delta U \) is:
\[ \delta U = \int_0^h \left\{ \int_{\Omega} \left[ \sum_{k=1}^N \left( \sigma_x \delta \varepsilon_x^0 + \sigma_y \delta \varepsilon_y^0 + \sigma_{xy} \delta \varepsilon_{xy}^0 + \gamma \delta \gamma \right) \right] \right\} \, d\varepsilon \, \varepsilon = 0 \] ... (7)

Substituting Eq. (4) into Eq. (7):
\[ \delta U = \int \left\{ N_x \delta \varepsilon_x^0 + N_y \delta \varepsilon_y^0 + N_{xy} \varepsilon_{xy}^0 + M_x \delta k_x^0 + M_y \delta k_y^0 + M_{xy} \delta k_{xy}^0 + M_x^0 \delta k_x^0 + M_y^0 \delta k_y^0 + M_{xy}^0 \delta k_{xy}^0 + Q_x \delta y_{xz}^0 + Q_y \delta y_{yz}^0 + Q_{xy} \delta y_{xz}^0 + Q_{yz} \delta y_{yz}^0 \right\} \, d\varepsilon \, \varepsilon = 0 \] ... (8)

- The virtual work done \( \delta V \) is:
\[ \delta V = -\int_A \left\{ N_x \delta \varepsilon_x^0 \right\} \frac{\partial^2 (w_{a} + w_{be} + w_{sh})}{\partial x^2} \, dx \, dy = 0 \] ... (11)
\[ \delta T = \int \frac{h}{2} \rho \left\{ \left[ \dot{u} - \frac{\partial w_{be}}{\partial x} + f \frac{\partial w_{sh}}{\partial x} \right] \frac{\partial \dot{u}}{\partial y} - \frac{\partial \dot{w}_{be}}{\partial x} - \frac{\partial \dot{w}_{sh}}{\partial x} - f \frac{2 \dot{w}_{sh}}{\partial x} \right\} \, dx \, dy \]

Fig. 1. coordinate system of laminated plates.
\[ z \frac{\partial^2 \tilde{w}_{be}}{\partial y^2} + f \frac{\partial^2 \tilde{w}_{sh}}{\partial y^2} + \left[ w_a + w_{be} + w_{sh} \right] [\tilde{\delta} \tilde{w}_a + \tilde{\delta} \tilde{w}_{be} + \tilde{\delta} \tilde{w}_{sh}] \text{d}v \]

\[ \delta T = \int \left[ \int \left( -I_1 \tilde{u} - I_2 \frac{\partial \tilde{w}_{be}}{\partial x} + I_3 \frac{\partial \tilde{w}_{sh}}{\partial x} \right) \text{d}x + \left( \tilde{\tilde{w}}_a + \tilde{\tilde{w}}_{be} + \tilde{\tilde{w}}_{sh} \right) \tilde{\delta} \tilde{w}_a \right] \text{d}y \]

\[ \left( -I_1 \tilde{v} + I_3 \frac{\partial \tilde{w}_{be}}{\partial x} - I_5 \frac{\partial \tilde{w}_{sh}}{\partial y} \right) \tilde{\delta} \tilde{w}_{be} + \left( \tilde{\tilde{w}}_a + \tilde{\tilde{w}}_{be} + \tilde{\tilde{w}}_{sh} \right) \tilde{\delta} \tilde{w}_{be} + \left( \tilde{\tilde{w}}_a + \tilde{\tilde{w}}_{be} + \tilde{\tilde{w}}_{sh} \right) \tilde{\delta} \tilde{w}_{sh} \right] \text{d}x \text{d}y \]

\[ \text{Where:} \]

\[ \left( I_1, I_2, I_3, I_4, I_5, I_6 \right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left( z, z^2, f(z), z f(z), [f(z)]^2 \right) \text{d}z \]

\[ \text{2.3 Equation of Motion} \]

The Euler-Lagrange is obtained by substituting equation (8 - 12) into equation (6), then setting the coefficient of $\left( \tilde{\delta}u, \tilde{\delta}v, \tilde{\delta}w_a, \tilde{\delta}w_{be}, \tilde{\delta}w_{sh} \right)$ of Eq.(6) to zero separately, this give five equations of motion as follows:

\[ \delta u : \frac{\partial^2 \tilde{X}}{\partial x^2} + \frac{\partial^2 \tilde{X}}{\partial y^2} = I_1 \tilde{u} \]

\[ \delta v : \frac{\partial^2 \tilde{X}}{\partial x \partial y} + \frac{\partial^2 \tilde{X}}{\partial y \partial x} = I_3 \tilde{v} \]

\[ \delta w_{be} : \frac{\partial^2 \tilde{X}}{\partial x^2} + \frac{2}{\partial x^2} \tilde{M} + \frac{2}{\partial y^2} \tilde{M} + N(w) = I_1 \left( \tilde{w}_a + \tilde{w}_{be} + \tilde{w}_{sh} \right) - I_3 \left( \frac{\partial \tilde{w}_{be}}{\partial x} + \frac{\partial \tilde{w}_{be}}{\partial y} \right) 
\]

\[ \delta w_{sh} : \frac{\partial^2 \tilde{X}}{\partial x^2} + \frac{2}{\partial x^2} \tilde{M} + \frac{2}{\partial y^2} \tilde{M} + \tilde{Q}_{xx} + \tilde{Q}_{yy} + \tilde{Q}_{xy} + \tilde{Q}_{yy} \]

\[ \left( \tilde{w}_a + \tilde{w}_{be} + \tilde{w}_{sh} \right) \right] \frac{\partial^2 \tilde{w}_a}{\partial x^2} \right] \frac{\partial^{2 \ast} \tilde{w}_{sh}}{\partial y^2} \bigg] \text{d}x \text{d}y \]

\[ \text{Where:} \]

\[ N(w) = N_x \left( \frac{\partial^2 (w_a + w_{be} + w_{sh})}{\partial x^2} + N_y \left( \frac{\partial^2 (w_a + w_{be} + w_{sh})}{\partial y^2} \right) + 2N_{xy} \left( \frac{\partial^2 (w_a + w_{be} + w_{sh})}{\partial x \partial y} \right) \right) \]

\[ \text{The result forces are given by: Reddy [9].} \]

\[ \left\{ \begin{array}{l}
N_x \\
N_y \\
N_{xy}
\end{array} \right\} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ \left\{ M_b \right\} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ \left\{ Q_x \right\} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ \left\{ Q_y \right\} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ Q_{xz} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ Q_{yz} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ Q_{xzz} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ Q_{yzz} = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} \text{d}z \]

\[ \text{The plane stress reduced stiffness } Q_{ij} \text{ is:} \]

\[ Q_{11} = \frac{E_1}{1-\nu_{12} \nu_{21}} = \frac{E_2}{1-\nu_{12} \nu_{21}} \]

\[ Q_{22} = \frac{E_2}{1-\nu_{12} \nu_{21}} \]

\[ Q_{66} = G_{12}, Q_{44} = G_{23}, Q_{55} = G_{13} \]

\[ \text{From the constitutive relation of } k^{th} \text{ layer lamina, the transformed stress-strain relation are:} \]

\[ \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} = \left\{ \begin{array}{l}
\frac{Q_{11}}{Q_{21}} \frac{Q_{12}}{Q_{22}} 0 0 0 \frac{Q_{11}}{Q_{21}} \frac{Q_{12}}{Q_{22}}
\end{array} \right\} \text{d}z \]

\[ \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} = \left\{ \begin{array}{l}
0 0 0 \frac{Q_{66}}{Q_{44}} \frac{Q_{66}}{Q_{44}}
\end{array} \right\} \text{d}z \]

\[ \left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} = \left\{ \begin{array}{l}
0 0 0 0 \frac{Q_{55}}{Q_{44}} \frac{Q_{55}}{Q_{44}}
\end{array} \right\} \text{d}z \]
\[
\begin{align*}
&M_{x} = \begin{bmatrix} B_{11}^{s} & B_{12}^{s} & B_{16}^{s} \\ B_{12}^{s} & B_{22}^{s} & B_{26}^{s} \\ B_{16}^{s} & B_{26}^{s} & B_{66}^{s} \end{bmatrix} \begin{bmatrix} \varepsilon_{x}^{0} \\ \varepsilon_{y}^{0} \\ \gamma_{xy} \end{bmatrix} \\
&M_{y} = \begin{bmatrix} D_{11}^{s} & D_{12}^{s} & D_{16}^{s} \\ D_{12}^{s} & D_{22}^{s} & D_{26}^{s} \\ D_{16}^{s} & D_{26}^{s} & D_{66}^{s} \end{bmatrix} \begin{bmatrix} k_{x}^{b} \\ k_{y}^{b} \\ k_{xy}^{b} \end{bmatrix} \\
&M_{xy} = \begin{bmatrix} H_{11}^{s} & H_{12}^{s} & H_{16}^{s} \\ H_{12}^{s} & H_{22}^{s} & H_{26}^{s} \\ H_{16}^{s} & H_{26}^{s} & H_{66}^{s} \end{bmatrix} \begin{bmatrix} k_{x}^{s} \\ k_{y}^{s} \\ k_{xy}^{s} \end{bmatrix}
\end{align*}
\]

Where:

\[
\begin{align*}
&Q_{yz}^{a} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{x}^{a} \\ \varepsilon_{y}^{a} \end{bmatrix} + \begin{bmatrix} A_{44}^{0} & A_{45}^{0} \\ A_{45}^{0} & A_{55}^{0} \end{bmatrix} \begin{bmatrix} \gamma_{xy}^{a} \\ \gamma_{yx}^{a} \end{bmatrix} \\
&Q_{xz}^{a} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{x}^{a} \\ \varepsilon_{y}^{a} \end{bmatrix} + \begin{bmatrix} A_{44}^{0} & A_{45}^{0} \\ A_{45}^{0} & A_{55}^{0} \end{bmatrix} \begin{bmatrix} \gamma_{xy}^{a} \\ \gamma_{yx}^{a} \end{bmatrix}
\end{align*}
\]

\[V = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn} \cos \alpha_{m} \sin \beta_{n} \cos \beta_{n} \sin \alpha_{m} \]

2.4 Navier’s Solution

To solve equations of motion (15-16), Navier’s generalized displacements are used which satisfy the boundary conditions of the problem as shown in Fig. 2, therefore Simply supported boundary conditions are satisfied by assuming the following form of displacements: Reddy [9]

\[U = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos \alpha_{m} \sin \beta_{n} \cos \beta_{n} \sin \alpha_{m} \]

Where:

\[\alpha = \frac{m \pi}{a}, \quad \beta = \frac{n \pi}{b}\]

and \((U_{mn}, V_{mn}, W_{bmn}, W_{shmn}, W_{ammn})\) are arbitrary constants.

The following stiffnesses are zero if the Navier solution exists,

\[\begin{align*}
A_{16} &= A_{26} = D_{16} = D_{26} = H_{16} = H_{26} = 0 \\
B_{12} &= B_{16} = B_{26} = B_{66} = B_{12}^{s} = B_{16}^{s} = B_{26}^{s} = 0 \\
A_{45} &= A_{45}^{0} = A_{45}^{s} = 0
\end{align*}\]

2.5 Vibration Analysis

Developing mass matrix and stiffness matrix from solution of homogeneous equations, when mechanical loading is equal to zero for free vibration, then eigenvalue equation is derived and the natural frequencies of vibration for simply supported plate are obtained.

\[\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 \\ s_{12} & s_{22} & s_{23} & s_{24} & 0 \\ s_{13} & s_{23} & s_{33} & s_{34} & 0 \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} \\ 0 & 0 & 0 & s_{45} & s_{55} \end{bmatrix} = \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{shmn} \\ W_{ammn} \end{bmatrix}
\]

\[\begin{bmatrix} m_{11} & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} & m_{35} \\ 0 & 0 & m_{43} & m_{44} & m_{45} \\ 0 & 0 & m_{51} & m_{51} & m_{55} \end{bmatrix} = \begin{bmatrix} \bar{U}_{mn} \\ \bar{V}_{mn} \\ \bar{W}_{bmn} \\ \bar{W}_{shmn} \\ \bar{W}_{ammn} \end{bmatrix}
\]

\[\|[S] - w^{2}[M]\| = 0\]

Where \([S_{ij}] = \text{stiffness matrix elements and } [M_{ij}] = \text{mass matrix}\).

2.6 Buckling

The applied loads for buckling analysis, are supposed to be in-plan forces
The fundamental natural frequency and critical load 
\[ N_{x}^{0} = N_{0}, \quad N_{y}^{0} = N_{0}, \quad \gamma = \frac{N_{x}^{0}}{N_{y}^{0}}, \quad N_{xy} = 0 \]

\[
\begin{bmatrix}
    s_{11} & s_{12} & s_{13} & s_{14} & 0 \\
    s_{12} & s_{22} & s_{23} & s_{24} & 0 \\
    s_{13} & s_{23} & s_{33} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{34} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & -N_{0}(\alpha^{2} + \gamma \beta^{2}) \\
    s_{14} & s_{24} & s_{34} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{44} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{45} - N_{0}(\alpha^{2} + \gamma \beta^{2}) \\
    0 & 0 & -N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{45} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{55} - N_{0}(\alpha^{2} + \gamma \beta^{2})
\end{bmatrix} \begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{bemm} \\
    W_{shmn} \\
    W_{amn}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

2.7 Free Vibration Analysis Under Initial Stress

The natural frequency is investigated with action of buckling, a ratio critical load (d) is

\[
\begin{bmatrix}
    s_{11} & s_{12} & s_{13} & s_{14} & 0 \\
    s_{12} & s_{22} & s_{23} & s_{24} & 0 \\
    s_{13} & s_{23} & s_{33} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{34} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & -N_{0}(\alpha^{2} + \gamma \beta^{2}) \\
    s_{14} & s_{24} & s_{34} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{44} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{45} - N_{0}(\alpha^{2} + \gamma \beta^{2}) \\
    0 & 0 & -N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{45} - N_{0}(\alpha^{2} + \gamma \beta^{2}) & s_{55} - N_{0}(\alpha^{2} + \gamma \beta^{2})
\end{bmatrix} \begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{bemm} \\
    W_{shmn} \\
    W_{amn}
\end{bmatrix} + \begin{bmatrix}
    m_{11} & 0 & 0 & 0 & 0 \\
    0 & m_{22} & 0 & 0 & 0 \\
    0 & 0 & m_{33} & m_{34} & m_{35} \\
    0 & 0 & m_{44} & m_{45} & m_{45} \\
    0 & 0 & m_{11} & m_{12} & m_{55}
\end{bmatrix} \begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{bemm} \\
    W_{shmn} \\
    W_{amn}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

3. Results and Discussion

3.1 Vibration and Buckling Results

The fundamental natural frequency and critical buckling for cross-ply plate with different design parameters for simply supported boundary condition, is analyzed and solved used MATLAB programming. We derive equation of motion depending on Refined plate theory using Navier solution to obtain vibration characteristic of plate under initial stress. To examine the validity of the derived equation and performance of computer programming for vibration and buckling stress of cross-ply laminated simply supported plate, a comparison with others researchers for different layers, thickness ratio (a/h) and orthotropy ratio (E1/E2). The non-dimensional natural frequency of antisymmetric cross-ply two, four, six and ten layer of thick plate (a/h=5) as a function of orthotropy ratio (E1/E2) shown in table (1) for the mechanical properties \[ G_{12} = G_{13} = 0.6 E_{2}, \quad G_{23} = 0.5 E_{2}, \quad v_{12} = 0.25 \] while Table (2) shows the Non-dimensional fundamental frequencies of antisymmetric square laminated plate for various values of thickness ratio and modulus ratio (E1/E2=40). The natural frequency shows good agreement with other researchers. The present theory is also close agreement with other theory for critical buckling as shown in table (3) which compared with [2] for Non-dimensional uniaxial buckling load of simply supported antisymmetric layer for (a/h=10), while Table (4) and Table (5) are compared with [9] which show the Non-dimensional uniaxial and biaxial buckling load of simply supported antisymmetric cross-ply for two and eight layers for various thickness ratio (a/h) and as a function of modulus ratios (E1/E2).

3.2 Vibration of Plate Under Initial Stress Results

Vibration analysis of present work is used but adding initial in-plane stress to investigate the validity of Refined theory for such case. Table (6) shows Non-dimensional natural frequency under various uniaxial loads ratio(d) for simply supported cross-ply [0/90/0] square plate with orthotropy ratio (E1/E2=10). Table (7) shows Non-dimensional natural frequency under various uniaxial loads ratio with various thickness ratio (a/h) for simply supported cross-ply [0/90/0] square plate with orthotropy ratio (E1/E2=40). The fundamental frequency decrease when increasing the value of compressive stress until the lowest
natural frequency vanished when inplane stress reaches the critical buckling stress, which proved by other researchers, as shown in Fig (3) and Fig (4).

Table 1,
Non-dimensional natural frequencies of square laminate with $a/h=5$, $G_{12} = G_{13} = 0.6 \ E_2$, $G_{23} = 0.5 \ E_2$, $\nu_{12} = 0.25$

<table>
<thead>
<tr>
<th>No of layers</th>
<th>source</th>
<th>$E_1/E_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>$(0/90)_2$</td>
<td>3D [10]</td>
<td>9.4055</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>9.632</td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>9.301</td>
</tr>
<tr>
<td>$(0/90)_3$</td>
<td>3D [10]</td>
<td>9.8398</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>9.925</td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>9.69</td>
</tr>
<tr>
<td>$(0/90)_5$</td>
<td>3D [10]</td>
<td>10.0843</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>10.074</td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>10.905</td>
</tr>
</tbody>
</table>

Table 2,
Non-dimensional natural frequencies of cross-ply square laminate $E_1/E_2=40.$

| No of layer | method        | $a/h$ |
|            |               | 10    | 20    | 50    | 100   |
| $(0/90)_1$ | TSDD [11]     | 10.56 | 11.10 | 11.27 | 11.30 |
|            | Present       | 10.55 | 11.10 | 11.27 | 11.30 |
| $(0/90)_2$ | TSDD [11]     | 14.84 | 16.57 | 17.18 | 17.27 |
|            | FSDT [12]     | 14.92 | 16.60 | 17.18 | 17.27 |
|            | PPT [4]       | 14.84 | 16.57 | 17.18 | 17.27 |
|            | Present       | 14.85 | 16.57 | 17.18 | 17.27 |
| $(0/90)_3$ | TSDD [11]     | 15.46 | 17.37 | 18.06 | 18.16 |
|            | FSDT [12]     | 15.50 | 17.39 | 18.06 | 18.17 |
|            | PPT [4]       | 15.46 | 17.37 | 18.06 | 18.16 |
|            | Present       | 15.47 | 17.38 | 18.07 | 18.18 |

Table 3,
Non-dimensional uniaxial buckling load of simply supported square laminates, $(a/h=10)$, $G_{12} = G_{13} = 0.6 \ E_2$, $G_{23} = 0.5 \ E_2$, $\nu_{12} = 0.25$

| No of layer | method   | $N$    | Diff |
|            |          |       |      |
| 4           | FSDT [12] | 22.806 | -    |
|             | PPT [3]   | 22.57  | 1.03 |
|             | Present   | 22.593 | 0.93 |
|             | ANSYS     | 22.134 | 2.94 |
| 6           | FSDT [12] | 24.5777 | -    |
|             | PPT [3]   | 24.4581 | 0.48 |
|             | Present   | 24.483 | 0.38 |
|             | ANSYS     | 23.78  | 3.2  |
| 10          | FSDT [12] | 25.45  | -    |
|             | PPT [3]   | 25.4225 | 0.10 |
|             | Present   | 25.44  | 0.03 |
|             | ANSYS     | 24.357 | 4.29 |
Table 4, 
Non-dimensional uniaxial buckling load of antisymmetric square laminates $G_{12} = G_{13} = 0.5E_2$ , $G_{23} = 0.2E_2$ , $v_{12} = 0.25$

<table>
<thead>
<tr>
<th>Method</th>
<th>$E_1/E_2=10$</th>
<th>$E_1/E_2=25$</th>
<th>$E_1/E_2=40$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>(0/90)</td>
<td>(0/90)</td>
<td>(0/90)</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td>5.792</td>
<td>9.186</td>
</tr>
<tr>
<td>20</td>
<td>(0/90)</td>
<td>(0/90)</td>
<td>(0/90)</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td>6.228</td>
<td>10.424</td>
</tr>
<tr>
<td>100</td>
<td>(0/90)</td>
<td>(0/90)</td>
<td>(0/90)</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td>6.382</td>
<td>10.895</td>
</tr>
</tbody>
</table>

Table 5, 
Non-dimensional biaxial buckling load of antisymmetric cross-ply laminates

<table>
<thead>
<tr>
<th>Method</th>
<th>$E_1/E_2=10$</th>
<th>$E_1/E_2=25$</th>
<th>$E_1/E_2=40$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a/h</td>
<td>(0/90)</td>
<td>(0/90)</td>
<td>(0/90)</td>
</tr>
<tr>
<td>10</td>
<td>(0/90)</td>
<td>[9]</td>
<td>2.873</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td></td>
<td>2.896</td>
</tr>
<tr>
<td>20</td>
<td>(0/90)</td>
<td>[9]</td>
<td>3.102</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td></td>
<td>3.114</td>
</tr>
<tr>
<td>100</td>
<td>(0/90)</td>
<td>[9]</td>
<td>3.184</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td></td>
<td>3.191</td>
</tr>
</tbody>
</table>

Table 6, 
Dimensionless natural frequency of a laminated plate under buckling different ratio (d).

<table>
<thead>
<tr>
<th>d</th>
<th>Method</th>
<th>Fundamental frequency ($\bar{\omega}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[13]</td>
<td>10.649</td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>10.645</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td>10.665</td>
</tr>
<tr>
<td>0.25</td>
<td>[13]</td>
<td>9.231</td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>9.231</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td>9.236</td>
</tr>
<tr>
<td>0.5</td>
<td>[13]</td>
<td>7.544</td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>7.544</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td>7.541</td>
</tr>
<tr>
<td>0.75</td>
<td>[13]</td>
<td>5.379</td>
</tr>
<tr>
<td></td>
<td>ANSYS</td>
<td>5.379</td>
</tr>
<tr>
<td></td>
<td>present</td>
<td>5.332</td>
</tr>
</tbody>
</table>
Table 7,
Non-dimensional natural frequency under various uniaxial loads ratio with various thickness ratio (a/h) for simply supported cross-ply [0/90/0] square plate and orthotropy ratio (E1/E2=40).

<table>
<thead>
<tr>
<th>Ratio of N_{cr}</th>
<th>a/h=5</th>
<th>a/h=10</th>
<th>a/h=50</th>
<th>a/h=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.176</td>
<td>14.78</td>
<td>18.585</td>
<td>18.759</td>
</tr>
<tr>
<td>0.1</td>
<td>9.654</td>
<td>14.02</td>
<td>17.632</td>
<td>17.79</td>
</tr>
<tr>
<td>0.2</td>
<td>9.102</td>
<td>13.22</td>
<td>16.623</td>
<td>16.77</td>
</tr>
<tr>
<td>0.3</td>
<td>8.514</td>
<td>12.36</td>
<td>15.550</td>
<td>15.96</td>
</tr>
<tr>
<td>0.4</td>
<td>7.882</td>
<td>11.44</td>
<td>14.39</td>
<td>14.35</td>
</tr>
<tr>
<td>0.5</td>
<td>7.195</td>
<td>10.45</td>
<td>13.142</td>
<td>13.26</td>
</tr>
<tr>
<td>0.6</td>
<td>6.436</td>
<td>9.34</td>
<td>11.754</td>
<td>11.86</td>
</tr>
<tr>
<td>0.7</td>
<td>5.574</td>
<td>8.09</td>
<td>10.179</td>
<td>10.27</td>
</tr>
<tr>
<td>0.8</td>
<td>4.551</td>
<td>6.61</td>
<td>8.311</td>
<td>8.38</td>
</tr>
<tr>
<td>0.9</td>
<td>3.218</td>
<td>4.67</td>
<td>5.877</td>
<td>5.93</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

4. Conclusion

Natural frequency and buckling stress of simply supported cross-ply square plate subject to initial axial stress have been obtained by using Refined plate theory. It is observed good results for natural frequency and critical buckling for uniaxial and biaxial load as compared with other researchers.

The following conclusions may be drawn from the present analysis:
1. Refined plate theory for analyzing natural frequency and buckling stresses of cross-ply square plate has been presented. It is observed that the natural frequency and buckling load increasing as the number of layer and thickness ratio increases.
2. The buckling stresses can be calculated through the stability equation as Eigen value problems. Another method to obtain the critical stress of cross-ply plate subject to axial and uniaxial in-plane stresses is to compute natural frequency by increasing the absolute value of compressive stress until the lowest natural frequency vanishes.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Discretion</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Plate dimension in x-direction</td>
<td>m</td>
</tr>
<tr>
<td>b</td>
<td>Plate dimension in y-direction</td>
<td>m</td>
</tr>
<tr>
<td>h</td>
<td>Plate thickness</td>
<td>m</td>
</tr>
<tr>
<td>A_{ij} , B_{ij} , D_{ij} , B_{ij}</td>
<td>Extension, bending extension coupling</td>
<td>N/m</td>
</tr>
<tr>
<td>E1, E2, E3</td>
<td>Elastic modulus components</td>
<td>GP</td>
</tr>
<tr>
<td>G_{23} , G_{12}</td>
<td>Shear modulus components</td>
<td>GP</td>
</tr>
</tbody>
</table>
5. References


تحليل الاهتزازات تحت تأثير الإجهاد الأولي باستخدام نظرية اللوحة المكررة

أبيهال عباس صادق

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الخلاصة

يتم الحصول على التردد الطبيعي تحت الضغوط الأولية للحصول على صفائح الرقائق المقاطعة لللوحة (Refined plate) باستخدام نظرية اللوحة طبقية المتماثلة (Navier) دون استخدام عوامل تصحيح اللوحة. يتم استخدام عوامل التصميم من عوامل التصميم والمحاسبة للمشكلة لـ (Eigen simply supported). يتم دراسة تأثير عدد من عوامل التصميم مثل نسبة العامل (E1 / E2) (a / h) (N) على التردد الطبيعي والضغط على الألواح المحيطة. يتم مقارنة النتائج مع بحثين آخرين.