



Free Vibration Analysis of Laminated Composite Plates with General Boundary Elastic Supports Under Initial Thermal Load

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Abstract

Free vibration behavior was developed under the ratio of critical buckling temperature of laminated composite thin plates with the general elastic boundary condition. The equations of motion were found based on classical laminated plate theory (CLPT) while the solution functions consists of trigonometric function and a continuous function that is added to guarantee the sufficient smoother of the so-named remaining displacement function at the boundaries, in this research, a modified Fourier series were used, a generalized procedure solution was developed using Ritz method combined with the imaginary spring technique. The influences of many design parameters such as angles of layers, aspect ratio, thickness ratio, and ratio of initial in-plane thermal load in addition to different boundary conditions on the natural frequencies of laminated plate is analyzed. In general, the changes of fundamental natural frequency is inversely proportional with the ratio of thermal buckling load, also most parameters aspect ratio effect on the natural frequency about 35 – 40%. The present results were compared with those obtained by other researchers, and show good agreement.

Keywords: Thermal Buckling, laminated composite plate, general boundary condition, ritz method.

1. Introduction

Composite materials are widely employed in many engineering disciplines applications in mechanical and civil engineering. Many have been researching free vibration analysis under initial thermal load of laminated composite thin and thick plates, but the research has been focused on some problems that include a few forms of boundary conditions because the rectangular plates have 55 forms of boundary condition therefore the solution procedure becomes very tedious when use of mode shapes as the basis functions, that's why the former researchers have been used at opposite edges of one pair of simply supported and clamped, so with general boundary conditions one may have to apply convergent methods like Rayleigh-Ritz, the chosen of suitable admissible functions is of high significance when applying the method of

Rayleigh-Ritz due to precision of the results, many investigations are worked about thermal buckling of laminated plate by using classical laminated plate theory but no one studied effect of general boundary conditions analytically.

Random free vibrations under the thermal loading of laminated composite plates with all boundary conditions was studied at [1], the randomness in lamina coefficients of thermal expansion and material properties are taking into consideration. Based on higher-order shear deformation theory the system equations have been derived, incorporating rotary inertia effects. For handling the random Eigen value problem, the finite element method is applied. The research transacted for together numerical and experimental examinations of the free vibration behavior under different moisture and temperature of composite laminated plates presented at [9]. The governing equations have been found based on first-order

shear deformation theory (FSDT) for the behavior structural of the composite laminated plates. Quantitative results are shown to present the parameters' influence of woven fiber laminate of (material, geometry, and lamination) on the free vibration of composite plates for varying moisture and temperature concentrations. The vibration and buckling of initially thermal stressed composite plates with material properties dependent on temperature was presented at [3]. Using the variation method, the governing equations have been found including the transverse shear deformation influences. The influences of different parameters on the vibration and buckling behaviors of laminated plates are investigated with respect to the material properties that are dependent on temperature. The distribution temperature is supposed to be linear and uniform on the plate in the transverse direction. [15] Focused on the Vibro-acoustic response and buckling of the clamped laminated composite plate induced by a harmonic concentrated force in the thermal environment. For acoustic response and vibration, the analytical solution has been derived for a fully clamped boundary condition. Meantime, the buckling temperatures and natural frequencies of the plate in the uniform temperature environment are also established by utilizing the first-order shear deformation theory (FOSDT) and classical laminate theory (CLT). [5] Presented the influence of hydrothermal conditions such as moisture and temperature on buckling load and free vibration frequency of laminated composite plates. To determine the critical load of the plate, the finite strip method based on first-order shear deformation theory (FOSDT) was applied to estimate the displacement field of each strip in the finite strip formulation. Linear shape functions and the Hermitian method were applied for in-plane and out-of-plane transverse direction and the trigonometric shape functions were applied in the longitudinal direction. The effect of a variable in material properties under different moisture and temperature on natural frequency and buckling capacity was estimated of plates with various biaxial loading and end conditions, as well as, the influence of the layer's delamination on natural frequency and buckling load of the plate was researched in various cases. [10] Presented the modal analysis of composite laminated structures subjected to thermal effect. A green-Lagrange nonlinear finite element (FE) model has been advanced according to the TSDT (third-order shear deformation theory) for the analysis.

Most of the research mentioned above has dealt with the composite plates within specific boundary

condition, but in this research, we will deal with the general boundary condition of composite plates. The present work investigates the free vibration analysis under effect of critical buckling temperature of laminated composite thin plate with general boundary condition, based on classical laminated plate theory and using Rayleigh-Ritz method, the displacement function as Fourier cosine series, plus, an arbitrary continuous function proposed by [15] for first time, so the effect of angle-ply orientation, boundary conditions, aspect ratio, and different composite materials properties are examined.

2. Theoretical Analysis

2.1 Classical Laminated Plate Theory

Governing differential equation of laminated thin plate based on classical laminated plate theory (CLPT), for natural vibration under thermal buckling load is written as; [4].

$$D_{11} \frac{\partial^4 w}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + N_x^T \frac{\partial^2 w}{\partial x^2} + N_y^T \frac{\partial^2 w}{\partial y^2} + 2N_{xy}^T \frac{\partial^2 w}{\partial x \partial y} + I_0 \frac{\partial^4 w}{\partial t^2} = 0 \quad \dots (2.1)$$

$$\{N^T\} = \sum_{k=1}^L \int_{h_{k-1}}^{h_k} [A] \{\alpha\}^k \Delta T dz \quad \dots (2.2)$$

Where: D_{ij} are bending stiffness, $\{\alpha\}$ is a vector of thermal expansion coefficient, $[A]$ is the Extension stiffness matrix, w is the frequency of the lamina vibrations. Introducing the moment of the inertia (I_0) per unit area of the laminate at point (x, y) ;

$$I_0 = \int_{-h_b}^{h_t} \rho dz = \sum_{K=1}^L \rho_k (z_k - z_{k-1}) \quad \dots (2.3)$$

Where L : the total number of layers constructed the plate,

k : denotes the layer number,

h : the thickness of the lamina,

z_k , and z_{k-1} : distances from the reference plane of the lamina to the two surfaces of the k^{th} ply.

In this study, uniform temperature distribution is taken into consideration.

2.2 Total Mechanical Energy

The definition of the total mechanical energy is the aggregation of its kinetic energy and potential energy of a particles being applied on by only conservatively by constant forces [11].

$$E = \Pi - E_c = Constant \quad \dots (2.4)$$

Where E: Total mechanical energy of a system,
 E_c: Total kinetic energy of the system,
 Π: Total strain energy of the system.

Potential energy of purely bending plate under in plane thermal load is [14];

$$\begin{aligned} \Pi &= \frac{1}{2} \int_0^b \int_0^a \left[D_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + D_{22} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right. \\ &+ D_{66} \left(\frac{2\partial^2 w}{\partial x \partial y} \right)^2 + 2 \left(D_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right) \left. \right] dx dy \\ &+ \frac{1}{2} \int_0^b \left[k_{x0} w^2 + K_{x0} \left(\frac{\partial w}{\partial x} \right)^2 \right]_{x=0} dy \\ &+ \frac{1}{2} \int_0^b \left[k_{x1} w^2 + K_{x1} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{x=a} dy \\ &+ \frac{1}{2} \int_0^a \left[k_{y0} w^2 + K_{y0} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=0} dx \\ &+ \frac{1}{2} \int_0^a \left[k_{y1} w^2 + K_{y1} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=b} dx \\ &- \frac{1}{2} \int_0^b \int_0^a \left[N_x^T \left(\frac{\partial w}{\partial x} \right)^2 + N_y^T \left(\frac{\partial w}{\partial y} \right)^2 \right. \\ &+ 2N_{xy}^T \left(\frac{\partial^2 w}{\partial x \partial y} \right) \left. \right] dx dy \quad \dots (2.5) \end{aligned}$$

$$\Delta E=0 \text{ or } E= Constant \quad \dots (2.6)$$

$$E_c = \frac{1}{2} \omega^2 \iint I_o w_o^2 dx dy \quad \dots (2.7)$$

2.3 Boundary Conditions

The twisting and bending shear forces can be expressed in displacement function as; [4].

$$M_x = -D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} \quad \dots (2.8)$$

$$M_y = -D_{22} \frac{\partial^2 w}{\partial y^2} - D_{12} \frac{\partial^2 w}{\partial x^2} \quad \dots (2.9)$$

$$M_{xy} = -2D_{66} \frac{\partial^2 w}{\partial x \partial y} \quad \dots (2.10)$$

$$Q_x = -D_{11} \frac{\partial^3 w}{\partial x^3} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial y^2 \partial x} \quad \dots (2.11)$$

$$Q_y = -D_{22} \frac{\partial^3 w}{\partial y^3} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} \quad \dots (2.12)$$

For rectangular plate restrained elastically, the boundary conditions are:

$$k_{x0} w = Q_x \quad K_{x0} \frac{\partial w}{\partial x} = -M_x \quad \dots (2.13)$$

at $x = 0$

$$k_{x1} w = -Q_x \quad K_{x1} \frac{\partial w}{\partial x} = M_x \quad \dots (2.14)$$

at $x = a$

$$k_{y0} w = Q_y \quad K_{y0} \frac{\partial w}{\partial y} = -M_y \quad \dots (2.15)$$

at $y = 0$

$$k_{y1} w = -Q_y \quad K_{y1} \frac{\partial w}{\partial y} = M_y \quad \dots (2.16)$$

at $y = b$

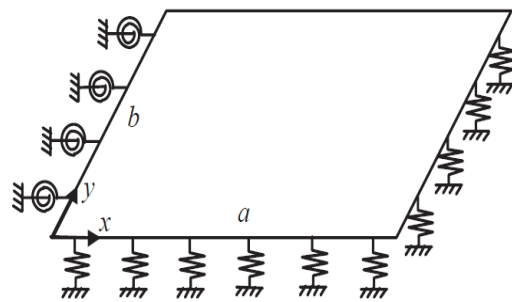


Fig. 1. A rectangular plate elastically restrained along edges, [15].

Where, K_{y0}, K_{y1} and K_{x0}, K_{x1} are the stiffness of the rotational spring, k_{y0}, k_{y1} and k_{x0}, k_{x1} are the stiffness of the translational spring, at $y = 0$ and b ($x = 0$ and a) respectively. Eqs. (13) - (16) shows a set of different B.Cs from which, the free boundary condition can be found by setting the spring constant to zero while setting the spring constant to infinity the clamped condition can be obtained (in actual calculation as a large number) and the simply supported can be obtained by setting the spring constant larger than zero and smaller than ∞ ($0 < K < \infty$), see (Fig. 1). From Eq. (8)-(16), the boundary conditions can be finally expressed as;

$$k_{x0} w = -D_{11} \frac{\partial^3 w}{\partial x^3} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial y^2 \partial x} \quad \dots (2.17)$$

$$k_{x1} w = D_{11} \frac{\partial^3 w}{\partial x^3} + (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial y^2 \partial x} \quad \dots (2.18)$$

$$K_{x0} \frac{\partial w}{\partial x} = D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2} \quad \dots (2.19)$$

$$K_{x1} \frac{\partial w}{\partial x} = -D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2} \quad \dots (2.20)$$

2.4 Admissible Functions

The permissible functions take a significant role in the Rayleigh-Ritz technique. The beam functions products are orderly selected as the permissible functions and the displacement function can be written as; [15]

$$w(x,y) = \sum_{m,n=1} A_{mn} X_m(x) Y_n(y) \quad (2.21)$$

Where $X_m(x)$, or $Y_n(y)$, is the characteristic beams functions contain the same B.Cs in the x-direction and y-direction, sequentially.

As a linear collection of trigonometric and hyperbolic functions, the functions of the beam can be broadly acquired, they contain a few unknown parameters and from which the boundary conditions are obtained. Accordingly, then, all boundary conditions essentially derived to various forms of beam functions. In veritable employment, this is plainly inappropriate, for a various boundary beam. An advanced method of Fourier series has been suggested for beams with a qualitative boundary at ends so as to avoid this obstacle, in which the characteristic functions are express in the term of; [19].

$$w(x) = \sum_{m=0}^{\infty} a_m \cos \lambda_{am} x + p(x) \left(\lambda_{am} = \frac{m\pi}{a} \right), 0 \leq x \leq a. \quad \dots (2.22)$$

Where $p(x)$ can be a count a qualitative continuous function that, whatever of B.Cs, is permanently elected to accept the subsequent equations [9]:

$$P''''(0) = W''''(0) = \alpha_0 \quad \dots (2.23)$$

$$P''''(a) = W''''(a) = \alpha_1 \quad \dots (2.24)$$

$$P'(0) = W'(0) = \beta_0 \quad \dots (2.25)$$

$$P'(a) = W'(a) = \beta_1 \quad \dots (2.26)$$

Still, as a continuous function that accepts Eq. (2.23)-(2.26), $P(x)$ formula doesn't consider the convergence of the series of the solution. So, the $P(x)$ function can be an option in different desired shape. As shown, postulate that $P(x)$ is a polynomial function,

$$P(x) = \sum_{n=0}^4 C_n P_n \left(\frac{x}{a} \right) \quad \dots (2.27)$$

Where $P_n(x)$ is the Legendre function C_n is the expansion constant of order n, the above expression for the function $P(x)$ can be express as;

$$P(x) = \zeta_a(x)^T \bar{\alpha} \quad \dots (2.28)$$

Where

$$\bar{\alpha} = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}^T \quad \dots (2.29)$$

And

$$\zeta_a(x)^T = \left\{ \begin{array}{l} -\frac{(15x^4 - 60ax^3 + 60a^2x^2 - 8a^4)}{360a} \\ \frac{(15x^4 - 30a^2x^2 + 7a^4)}{360a} \\ \frac{(6ax - 2a^2 - 3x^2)}{6a} \\ \frac{6a}{(3x^2 - a^2)} \end{array} \right\} \dots (2.30)$$

The results in Eq. (2.28)-(2.30) are obtained from much more direct but general approaches, [15].

So as to find the unknown constants of boundary, $\alpha_0, \alpha_1, \beta_0$, and β_1 , substitution of Eq. (2.22) and (2.28) into the boundary conditions Eq. (2.17)-(2.20) results in:

$$\bar{\alpha} = \sum_{m=0}^{\infty} H_a^{-1} Q_{am} a_m \quad \dots (2.31)$$

Where

$$H_a = \begin{bmatrix} 1 + \frac{8k_{x0}a^3}{360D_{11}} & \frac{7k_{x0}a^3}{360D_{11}} & \frac{-k_{x0}a}{3D_{11}} & \frac{-k_{x0}a}{6} \\ \frac{7k_{x1}a^3}{360D_{11}} & 1 + \frac{8k_{x1}a^3}{360D_{11}} & \frac{-k_{x1}a}{3D_{11}} & \frac{-k_{x1}a}{6} \\ \frac{a}{3} & \frac{a}{6} & \frac{K_{x0}}{D_{11}} + \frac{1}{a} & \frac{-1}{a} \\ \frac{a}{6} & \frac{a}{3} & \frac{-1}{a} & \frac{K_{x1}}{D_{11}} + \frac{1}{a} \end{bmatrix} \quad \dots (2.32)$$

And

$$Q_{am} = \left\{ (-1) \frac{k_{x0}}{D_{11}} \quad (-1)^m \frac{k_{x1}}{D_{11}} \quad -\lambda_{am}^2 \quad (-1)^m \lambda_{am}^2 \right\}^T \quad \dots (2.33)$$

For a totally free beam, it must be reminded that a matrix H_a becomes single. Via, this case can be controlled to a few extension by imaginary deliver springs to the edges of a beam with the littlest stiffness. It has been exhibited in [19]. In such remediation via the matrix might be unconditional. Although, the functions are much more appropriate for a specific condition and can be instantly applied in the method of Rayleigh-Ritz as the permissible functions.

By applying Eqs. (2.28) and (2.31), Eq. (2.22) can be expressed as

$$w(x) = \sum_{m=0}^{\infty} a_m \varphi_m^a(x) \quad \dots (2.34)$$

Where

$$\varphi_m^a(x) = \cos \lambda_{am} x + \zeta_a(x) H_a^{-1} Q_{am} \quad \dots (2.35)$$

Mathematically, Eq. (2.34) suggests that all of the functions of the beam shown as a function in the practical area with the principal functions $\{\varphi_m^a(x): m = 0, 1, 2, \dots \dots \dots\}$. Therefore, Eq. (2.21) expressed as:

$$w(x,y) = \sum_{m,n=0}^{\infty} A_{mn} \varphi_m^a(x) \varphi_n^b(y) \quad \dots (2.36)$$

Where

$$\varphi_n^b(y) = \cos \lambda_{bn}y + \zeta_b(y)H_b^{-1}Q_{bn} \quad \dots (2.37)$$

The terms for $\zeta_b(y), H_b$ and Q_{bn} can be, correspondingly, found from Eqs. (2.30), (2.32) and (2.33) by little shifting the x - concerning parameters by the y - concerning.

2.5 Determination of the Natural Frequency Under in-Plane Thermal Buckling Load

Substituting Eqs. (2.5) and (2.7) in Eq. (2.4), the total mechanical energy can be written in the following expressions:

$$\begin{aligned} E &= \frac{1}{2} \int_0^b \int_0^a \left[D_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + D_{22} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right. \\ &+ D_{66} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + 2 \left(D_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right) \Big] dx dy \\ &+ \frac{1}{2} \int_0^b \left[k_{x0} w^2 + K_{x0} \left(\frac{\partial w}{\partial x} \right)^2 \right]_{x=0} dy \\ &+ \frac{1}{2} \int_0^b \left[k_{x1} w^2 + K_{x1} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{x=a} dy \\ &+ \frac{1}{2} \int_0^a \left[k_{y0} w^2 + K_{y0} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=0} dx \\ &+ \frac{1}{2} \int_0^a \left[k_{y1} w^2 + K_{y1} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=b} dx \\ &- \frac{1}{2} \int_0^b \int_0^a \left[N_x^T \left(\frac{\partial w}{\partial x} \right)^2 + N_y^T \left(\frac{\partial w}{\partial y} \right)^2 \right] dx dy \\ &- \frac{1}{2} \omega^2 \iint I_o w^2 dx dy \quad \dots (2.38) \end{aligned}$$

Where the transverse displacement (w_o) is substituted as mentioned in section (2.4).

To calculate the natural frequency under in plane thermal buckling load action; N_x^T is left as a known ratio of the critical thermal buckling load N_{cr}^T . Performing the required mathematical processes (integrations and differentiations) of Eq. (2.38) and using Ritz method we get:

$$\frac{\partial E}{\partial A_{mn}} = 0 \quad \dots (2.39)$$

Eq. (2.39) gives homogenous equations as follow:

$$\begin{aligned} f(A_{mn}, N_{xx}^T, \omega) &= 0 \text{ for vibration under intial thermal stress} \\ &\dots (2.40) \end{aligned}$$

Solving Eq. (2.40) as an Eigen-value problem which is written as below:

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,(m^*n)} \\ \vdots & \ddots & \vdots \\ a_{(m^*n),1} & \dots & a_{(m^*n),(m^*n)} \end{bmatrix} \begin{Bmatrix} A_{11} \\ \vdots \\ A_{mn} \end{Bmatrix} = 0 \quad \dots (2.41)$$

Where a_{ij} are the coefficients of the nonzero unknowns A_{mn} . Finding the determinant of the first term of Eq. (2.41) and equating it to zero will lead to get the natural frequencies ω under intial thermal stress N_{cr}^T When M and N are more than 1, the natural frequencies ω under intial thermal stress N_{cr}^T is determined by solving the Eigen value problem. For different edge conditions and M and N greater than 1, the solution becomes more difficult and needs computer programming to find natural frequencies. In this study MATLAB R2017b is used to numerically solve the Eigen value problem to find the natural frequency under thermal buckling action.

3. Numerical Result

The natural frequency under the ratio of thermal buckling of composite laminated plate with the general elastic boundary condition is analyzed and solved using MATLAB R2017b. To verify the derived equations and performance of computer programming for vibration analysis of composite laminated plate, the numerical results of composite laminated plate were compared with those obtained by other researchers and those obtained by numerical program ANSYS. The frequencies are investigated with the effect of the in-plane thermal loading which is ratio of critical thermal buckling load. Several considerations are presented to investigate the frequency. In case of in-plane action, a ratio (d) of critical thermal load is applied. The effective ratio (d) is studied to present the behavior of the plate and its frequency. The dimensionless fundamental frequency for all edges simply supported symmetric cross-ply [0/90/0] square plates under two sets of thermal loading conditions with various side-to thickness ratios are obtained and presented in Table 1 along with those available in the [1], used a finite element method based on higher order shear deformation theory. From the table, as expected natural frequencies decrease with increasing temperature and thickness ratio due to reduced plate stiffness it can also be seen that the present results are in good agreement. The material properties are:

E_1	E_2	$G_{13}=G_{12}$	G_{23}
$40 \cdot E_2$	6.92Gpa	$0.6 \cdot E_2$	$0.5 \cdot E_2$
ν_{12}	α_1		α_2
0.25	$1.14 \cdot 10^{-6}/^\circ\text{K}$		$11.4 \cdot 10^{-6}/^\circ\text{K}$

Table 1,

Dimensionless first mode frequency ($\bar{\omega} = \omega_o a^2 \sqrt{\rho_c/E_2/h}$), for [0 90 0] plates under different load temperature)

Temp.	Present		[1]		Discrepancy %	
	a/h=20	a/h=50	a/h=20	a/h=50	a/h=20	a/h=50
	$\Delta T_{cr}=1401.64$	$\Delta T_{cr}=223.451$				
0	18.8145	18.9	17.483	18.7871	7.077	0.656
100	18.1817	14.456	17.172	16.2855	5.5534	11.234

Also obtained natural frequencies under different load temperature are compared with the results that obtained by [17], which they applied the classical laminate theory (CLT) and first order shear deformation theory (FOSDT) as listed in Table 2. As expected natural frequencies decrease as the temperature increase due to reduction of

plate stiffness, a rectangular laminated plate with dimensions $600 \times 400 \times 5$ mm³ is considered. The plate is composed of five layers with equal thickness. The orientations of the layer are [0/90/0/90/0], it can also be seen that the present results are close to results from Ref. [17]. The material property of each single layer is:

E_1	E_2	$G_{13}=G_{12}$	G_{23}
132Gpa	10.3Gpa	6.5Gpa	3.91Gpa
ν_{12}	α_1	α_2	ρ
0.3	$1.2 \cdot 10^{-6}/^\circ\text{C}$	$2.4 \cdot 10^{-6}/^\circ\text{C}$	1570 kg/m ³

Table 2,

Comparisons of natural frequencies (Hz) at different temperatures, $\Delta T_{cr}=95.124$.

Temp	0	20	40	60	80
Present	201.65	179.54	154.131	123.6	82.46
[17]	200.02	176.91	149.89	116.19	66.1
Discrepancy %	0.81	1.465	2.75	6	19.84

Fundamental natural frequency of square plates with various boundary conditions under different thermal buckling load ratio (d), is calculated and listed in Tables 3 and 4 for symmetric cross laminated plates and anti-symmetric angle ply which give small error when compared with results obtained by numerical program ANSYS. Stiffness of the plate with the clamped along two or four edges was greater than any other boundaries, therefore it vibrates with a higher frequency. In other hand, the frequency of the SFSF plate is minimal due to low stiffness. It is clear that the frequency was less than frequency that found without loading because of the reduced stiffness due to thermal loading.

The aspect and modulus ratios schemes of laminated plate for simply supported are changed in Table 5, the results of the frequency under thermal load ratio (d=0.5) and compared with those obtained by numerical program ANSYS, it can be noted that the frequency decreases with increasing the aspect and increases when the modulus ratio increases, due to stiffness changes.

Figure (2), shows the first four mode shapes of the laminated plate for CFFF symmetric cross ply in the reference temperature environment of the same materials properties in tables 3 and 5.

Table 3,

Dimensionless first mode frequency ($\bar{\omega} = \omega_0 a^2 \sqrt{\rho_c/E_2/h}$), for [0 90 90 0] plates under different load ratios of critical buckling temperature for different boundary conditions, ($E_1/E_2 = 15, G_{12} = 0.5E_2, G_{23} = 0.3356E_2, \nu_{12} = 0.3, \alpha_1 = 0.015e - 6, \alpha_2 = 1e - 6, a = b, \frac{a}{h} = 100$).

d	References	CCCC	SSSS	SCSC	CCCF	SSSF	SFSF	CFCF
0	MATLAB	26.4	15.53	24.26	12	6.87	4.1	9.283
	ANSYS	25.63	15.66	23.54	11.5	6.675	3.632	8.671
	Discrepancy %	3	0.83	3	4.167	2.84	11.4	6.6
0.25	MATLAB	22.86	13.45	21.33	10.38	6	3.537	8.04
	ANSYS	22.06	13	20.5	10.4	5.86	3	7.56
	Discrepancy %	3.5	3.346	3.89	0.2	2.33334	15.2	6
0.5	MATLAB	18.6676	11	18	8.5	4.86	2.8878	6.56
	ANSYS	17.68	10.353	17	9.2	5.0845	2.37	6.24
	Discrepancy %	5.3	5.882	5.556	7.61	4.4	18	4.88
0.75	MATLAB	13.2	7.76	13.706	6.02	3.44	2.042	4.6415
	ANSYS	11.4344	6.86	12.35	7.7455	4.1642	1.506	4.4672
	Discrepancy %	12	11.6	10	22.277	17.4	26.25	3.77

Table 4,

Dimensionless first mode frequency ($\bar{\omega} = \omega_0 a^2 \sqrt{\rho_c/E_2/h}$), for [30/-30]₄ plates under different load ratios of critical buckling temperature, ($E_1/E_2 = 15, G_{12} = 0.5E_2, G_{23} = 0.3356E_2, \nu_{12} = 0.3, \alpha_1 = 0.015e - 6, \alpha_2 = 1e - 6, a = b, \frac{a}{h} = 100$).

d	References	CCCC	SSSS	SCSC	CCCF	SSSF	SFSF	CFCF
0	MATLAB	27	12.5	25.02	12	5.3	4.81	11
	ANSYS	26.673	12.43	24.67	11.865	5.28	4.8	10.544
	Discrepancy %	1.2	0.56	1.4	1.125	0.377	0.208	4.1455
0.25	MATLAB	23.563	10.838	23.6	10.4	4.6	4.67	10.07
	ANSYS	23.124	10.75	23.32	10.8	4.57	4.25	10
	Discrepancy %	1.863	0.812	1.186	3.7	0.65	9	0.7
0.5	MATLAB	19.55	8.86	22.2	8.55	3.76	3.65	8.48
	ANSYS	18.773	8.757	21.88	9.6	3.7	3.63	9.056
	Discrepancy %	3.94	1.16	1.44	11	1.6	0.548	6.36
0.75	MATLAB	14.46	6.2876	20.7	6.1	2.667	3	6
	ANSYS	12.653	6.15	20.34	8.2	2.61	2.876	7.97
	Discrepancy %	12.5	2.2	1.74	25.5	2.14	4.127	24.7

Table 5,

Dimensionless first mode frequency ($\bar{\omega} = \omega_0 a^2 \sqrt{\rho_c/E_2/h}$), for [0 90 90 0] (SSSS) plates with effect of aspect and modulus ratios, ($G_{12} = 0.5E_2, G_{23} = 0.3356E_2, \nu_{12} = 0.3, \alpha_1 = 0.015e - 6, \alpha_2 = 1e - 6, \frac{a}{h} = 100$) natural frequency under load ratios of critical buckling temperature d=0.5).

References	a/b	E ₁ /E ₂ 5	10	15
MATLAB	0.5	19.848	28.27	34.72
ANSYS		18.77	26.76	33
Discrepancy %		5.43	5.34	5
MATLAB	1	5.88	7.4	8.654
ANSYS		6	7.5	8.74
Discrepancy %		2	1.33334	1
MATLAB	1.5	3.814	4.54	5.17
ANSYS		3.84	4.553	5.1
Discrepancy %		0.677	0.2855	1.354

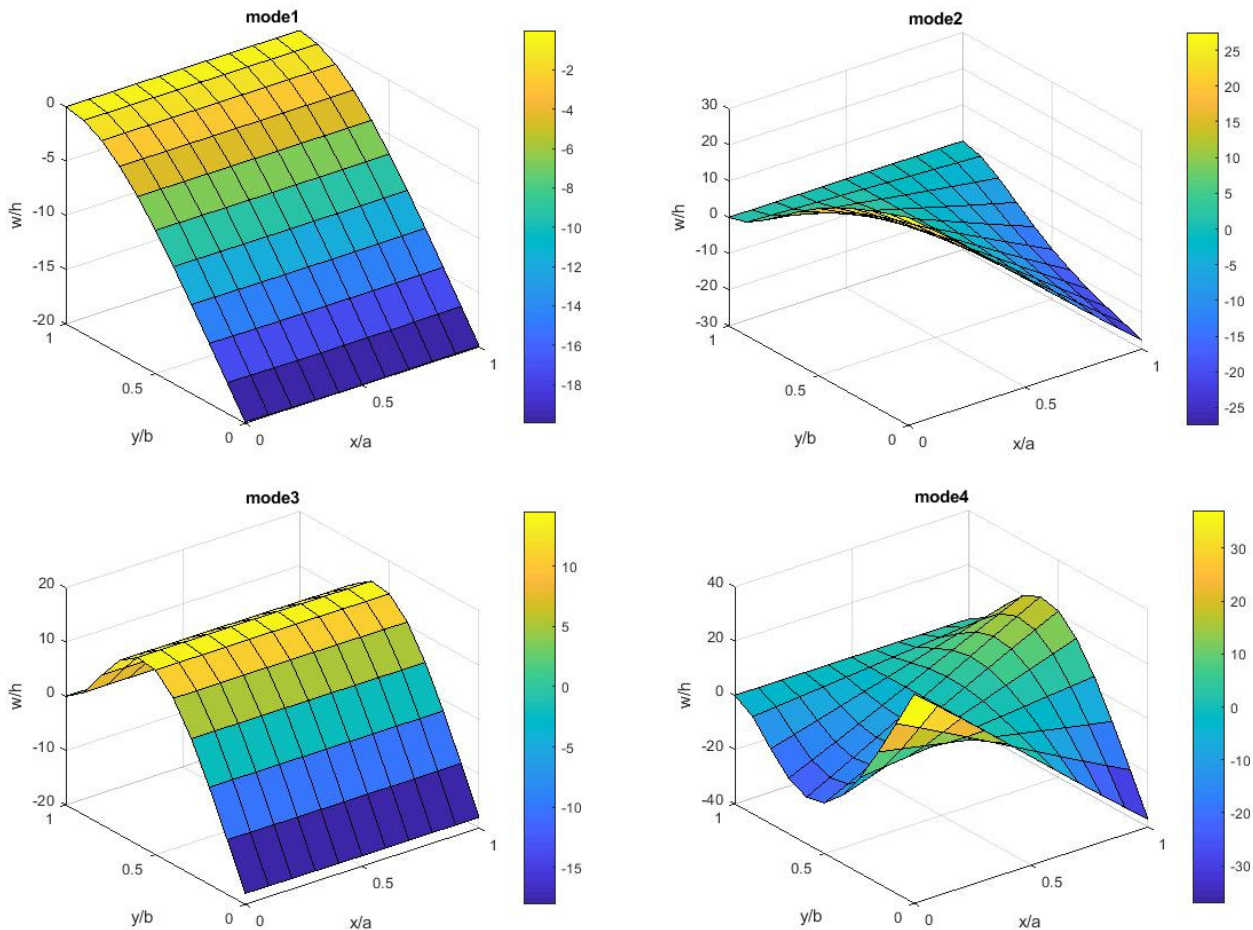


Fig. 2. Mode shape for free vibration of (CFFF) for (0/90/90/0) laminated square plate.

4. Conclusion

In the present work natural frequency of laminated plate with different angle schemes under ratios of thermal buckling load and general boundary conditions was developed using Ritz method based on a permissible function for the first time. Changing some design parameters such as thickness ratio, aspect ratio and orthotropic ratio are also studied, as expected fundamental frequency of the plate is affected by the ratio of initial thermal buckling load. The results present to this conclusions; the natural frequencies for laminates increase with the increase in aspect ratio and modules ratio E_1/E_2 . The reduction of natural frequencies in angle ply laminates is much higher compared to cross ply laminate. The SSSS plate is the most sensitive against simultaneous change in the system properties, and thermal expansion coefficients, while the CCCC plate is the least sensitive (Clamped edges conditions offer high stiffness). Therefore, this investigation showed that this function can be used to find work natural

frequency of laminated plate with different angle schemes under ratios of thermal buckling load and general boundary conditions.

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تحليل الاهتزاز الحر للصفائح المركبة ذات الدعامات المرنة ذات الحدود العامة تحت الحمل الحراري الأولي

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الخلاصة

يتم تطوير سلوك الاهتزاز الحر في ظل نسبة درجة حرارة الانبعاج للوحات مغلقة رقيقة مركبة مع شرط الحدود المرنة العامة. تم اعتماد معادلة الحركة على نظرية الألواح الرقائقي الكلاسيكية (CLPT) بينما تتكون وظائف الحل من دالة مثلثية ووظيفة مستمرة تضاف لضمان سلاسة كافية لوظيفة الإزاحة المتبقية المسماة على الحدود، في هذا البحث، تم استخدام سلسلة Fourier المعدلة، كما تم تطوير حل إجراء معمم باستخدام طريقة Ritz جنباً إلى جنب مع تقنية النابض التخيلية. مع تحليل تأثير العديد من معاملات التصميم مثل نسبة السمك، ونسبة العرض إلى الارتفاع، وزوايا الطبقات، ونسبة الحمل الحراري الأولي في المستوى فضلاً عن الظروف الحدودية المختلفة، على الترددات الطبيعية للصفائح الرقائقي. وتمت مقارنة النتائج الحالية مع تلك التي حصلنا عليها من قبل باحثين آخرين، واعطت اتفاقاً جيداً.